Intro to dimension Theory

For background and motivation, read Chapter 8 of Eisenbud.

Def: The <u>Krull dimension</u> (or just <u>dimension</u>) of a ring R, $\dim R$, is the supremum of lengths of chains of prime ideals in R.

e.g. the length of $P_r \supsetneq P_{r-1} \supsetneq \cdots \supsetneq P_0$ is r.

Ex: We will see later that in $\mathbb{C}[x_1, \dots, x_n]$,

$$(x_1, \dots, x_n) \supsetneq (x_1, \dots, x_{n-1}) \supsetneq \cdots \supsetneq (x_1) \supsetneq (0)$$

is a chain of maximal length, so $\dim \mathbb{C}[x_1, \dots, x_n] = n$.

Def: If $I \subsetneq R$ is an ideal, we define the <u>dimension</u> of I to be $\dim I := \dim R/I$.

If, in addition, I is prime, we define the <u>codimension</u> of I to be the supremum of lengths of chains of primes descending from I. Note that $\operatorname{codim} I = \dim R_I$.

If I is not prime, define $\operatorname{codim} I := \min\{\operatorname{codim} P \mid P \supseteq I \text{ is prime}\}$

We can extend the notion of dimension to modules:

Def: Let M be an R-module. The dimension of M is $\dim M := \dim \operatorname{ann} M = \dim R/\operatorname{ann} M$.

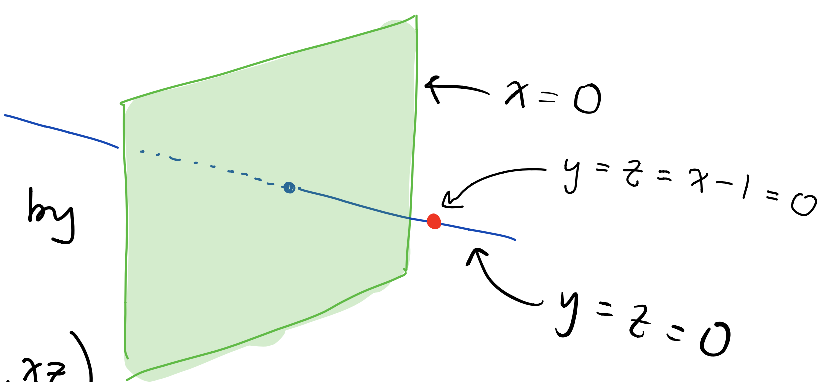
This can be confusing. If we consider $I \subsetneq R$ an ideal, where R is a domain, then $\dim I$, where we consider I as an R-module, is $\dim R/0 = \dim R$. So when $I \subsetneq R$, if we write $\dim I$, we mean the dimension as an ideal. Usually, it should be clear from context which we mean.

Question: Let $I \subsetneq R$ be an ideal. Why can't we define $\operatorname{codim} I = \dim R - \dim I$?

Answer: We'll see that this is true in "nice" cases (e.g. if R is a domain, finitely generated as a k-algebra), but it's not true more generally:

Ex: Consider the following picture in $\mathbb{A}^3$:

In $k[x,y,z]$, this is cut out by the ideal $(x)(y,z) = (xy, xz)$.



Consider $R = k[x,y,z]/(xy, xz)$.

Then $(x) \subsetneq (x,y) \subsetneq (x,y,z)$ are prime, so $\dim R \geq 2$. In fact $\dim R = 2$.

However, if we set $I = (x-1, y, z) \subsetneq R$, then

$\dim I = \dim R/I = 0$, since I is maximal.

But $\operatorname{codim} I = \dim R_I = \dim {k[x,y,z]_I}/{(y,z)} = \dim k[x]_{(x-1)} = 1.$

so $\dim I + \operatorname{codim} I \neq \dim R$.

The intuition is that $\dim R$ is the dimension of the largest component, whereas $\operatorname{codim} I$ gives the "local" codimension, i.e. the codimension of I in the component in which it lies.

Connection to Artinian rings

Recall: a ring R is Artinian if every strictly decreasing chain of ideals is finite. We proved the following near the beginning of the semester:

Thm: Let R be a ring. R is Artinian if and only if R is Noetherian and all its prime ideals are maximal.

Moreover, if R is Artinian it has only finitely many maximal ideals.

We deduced the following statement about Spec$R$:

Cor: If R is Noetherian, then R is Artinian $\Leftrightarrow$ Spec(R) is finite.

$\dim R = 0 \Leftrightarrow$ all its prime ideals are maximal, so we can translate these into a statement about dimension:

Cor: If R is Noetherian, then $\dim R = 0 \Leftrightarrow R$ is Artinian $\Leftrightarrow$ Spec$R$ is finite + discrete

Dimension and morphisms

Recall that the "going up" Theorem tells us that we can lift an increasing chain of prime ideals in R to a ring S integral over R. We also showed that if two prime ideals in S, one contained in the other, have the same intersection in R, then they must be equal.

We can use these theorems to compare the dimensions of ideals in R and ideals in S:

Prop: Let $\varphi: R \to S$ be a map of rings that makes S integral over R. Then any prime ideal of R containing $\ker \varphi$ is the preimage of some prime ideal of S.

Moreover, if $I \subseteq S$ is an ideal, then $\dim I = \dim \varphi^{-1}(I)$.

Pf: Note that if $I \subseteq S$, then $\ker \varphi \subseteq \varphi^{-1}(I)$, so

$R/\varphi^{-1}(I) \cong \varphi(R)/\varphi(R) \cap I$. Thus

$$\dim \varphi^{-1}(I) = \dim \left(R/\varphi^{-1}(I)\right) = \dim \varphi(R)/\varphi(R) \cap I.$$

So we can replace R with its image in S and assume $R \subseteq S$.

Then the first part of the statement follows immediately from going up.

For the second part, consider a chain of primes containing $\varphi^{-1}(I)$, $P_0 \subsetneq P_1 \subsetneq \cdots$. By going up, we can find $Q_0 \subsetneq Q_1 \subsetneq \cdots$, primes in S containing I s.t. $Q_i \cap R = P_i$. Thus $\dim I \geq \dim \varphi^{-1}(I)$.

If we start w/ a chain $Q_0 \subsetneq Q_1 \subsetneq \cdots$ containing I, then $Q_i \cap R \neq Q_j \cap R$ for $i \neq j$ by incomparability, so $\dim \varphi^{-1}(I) \geq \dim I$. Thus, equality holds. $\square$

We can also interpret this geometrically:

Cor: Let $\varphi: R \to S$ be a map of rings s.t. S is Noetherian and integral over R. Let $\varphi: \operatorname{Spec} S \to \operatorname{Spec} R$ be the corresponding map on Spec. Then

1.) The fibers of φ over closed points are finite.

2.) If $\overset{V(I)}{\overset{\shortparallel}{X}} \subseteq \operatorname{Spec} S$ is Zariski closed, then $\varphi(X) \subseteq \operatorname{Spec} R$ is a Zariski closed subset w/ the same dimension as X. That is, $\varphi(X) = V(J)$ and $\dim(I) = \dim(J)$.

Pf: WLOG, we can replace $\operatorname{Spec} S$ w/ X and $\operatorname{Spec} R$ w/ the closure of $\varphi(\operatorname{Spec} S)$ (ie. the smallest closed set containing it).

For 2.), we must show φ is surjective and S and R have the same dimension. If $P \in \operatorname{Spec} R$ is in the image of φ, then $P = \varphi^{-1}(Q)$, some $Q \in \operatorname{Spec} R$. In particular, $\ker \Psi \subseteq P$.

Thus, $V(\ker \Psi)$ is a closed set in $\operatorname{Spec} R$ containing the image of φ, so $V(\ker \Psi) = \operatorname{Spec} R$.

So for $P \in \operatorname{Spec} R$, $\ker \Psi \subseteq P$, so we can apply the prop above which says $\exists$ $Q \in \operatorname{Spec} S$ s.t. $P = \Psi^{-1}(Q) = \varphi(Q)$, so φ is surjective.

Also, $\dim S = \dim_S 0 = \dim_R \Psi^{-1}(0) = \dim_R \ker \Psi = \dim R$.

For 2.), let $m \in \operatorname{Spec} R$ be a closed point (i.e. a maximal ideal. Then $\varphi^{-1}(m) := \{Q \text{ prime in } S \text{ s.t. } \varphi^{-1}(Q) = m\}$ is some closed set $\underset{V(\cap Q)}{\overset{\shortparallel}{V(I)}} \subseteq \operatorname{Spec} S$. Thus,

$$\dim V(I) := \dim I = \dim \underset{m}{\underset{\shortparallel}{\varphi^{-1}(I)}} = 0.$$

$(Q \cap \varphi(R) = \varphi(m))$

Thus, the dimension of the fiber is 0, which means it's finite. $\square$